BACKTESTING RISK MEASURES

In particular Value at Risk and Expected Shortfall

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AGENDA

- What does backtesting mean?
- The Basel VaR vs ES dilemma
  - And the importance of backtestability
- VaR and ES: a closer look
  - Getting an intuition of why VaR is backtestable and ES (maybe) not
- Elicitability and backtestability
  - Formalizing intuition
- L'ES est mort, vive l'ES
  - How to backtest ES, nonetheless
- Did VaR create ES?
  - Indirect backtestability

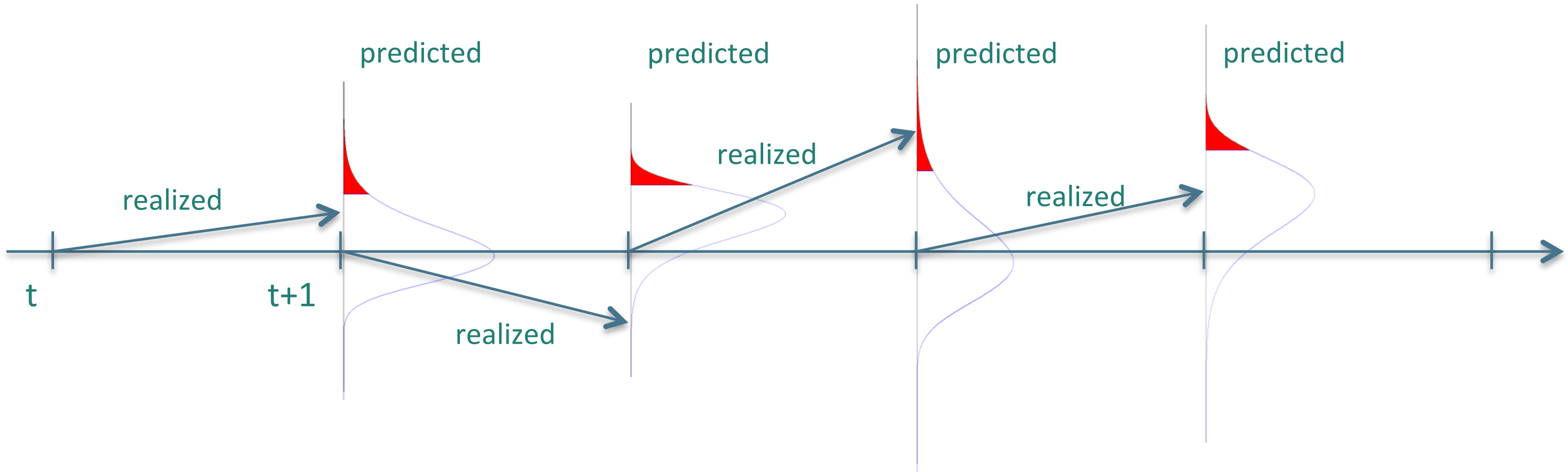
WHAT DOES BACKTESTING MEAN?

The Importance of Model Validation

TESTING PREDICTIONS

- **Setting**: a sequence of predictions on future events
- **Objective**: testing 'ex post' how good your predictions have been
- Ex: every week you forecast the winning horse for next Sunday's races
  - At the end of the season you can easily score your 'prediction model'
  - Why? Because the winning horse is publicly declared every week
- Ex: Banks have to forecast their risks and allocate capital accordingly
  - Risk models need to be validated -> backtested
  - But risk is not declared ex post
    - Is it always possible to backtest a risk measure?
    - What makes a measure backtestable or not?

WHAT DOES BACKTESTING MEAN?



- Backtesting a risk measure means testing forecasts against realizations. However
  - Distributions and risk measures do not materialize ex post like winning horses
  - Only one scenario at a time is revealed

- A risk measure is commonly said *backtestable* if there exists an observable test statistic, that allows to say whether predictions are over/under-estimated
  - E.g.: risk measure = VaR; test variable = VaR breaches counting

- Not all risk measures appear to be backtestable
  - However, a formal definition of 'backtestable' has long been missing

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THE BASEL VAR - ES DILEMMA

And the Importance of Backtesting

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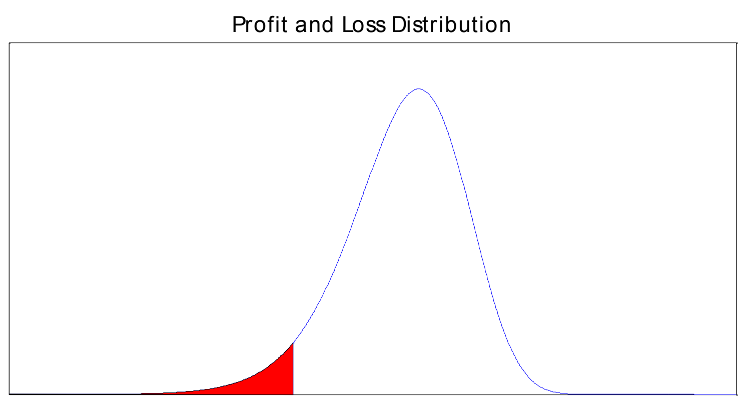
VAR AND ES

- VaR: the best of worst x% losses; → threshold of x% losses
- ES: the average of worst x% losses → expected x% loss



- ES
  - Multiple advantages: tail sensitivity, subadditivity (→ coherent), mathematical tractability, uniqueness, uses same risk models …
  - One drawback: how to backtest ES is still an open issue

BASEL REGULATION: VAR OR ES?

- 1994: RiskMetrics Technical Document popularizes "Value at Risk" (VaR)
- 1996: Basel Committee internal-based approach to capital adequacy, based on VaR
- 1997: Artzner et al. "Coherent Measures of Risk": axioms for sensible risk measures. VaR criticized for not complying
- 2001: Rockafellar and Uryasev, Acerbi and Tasche, define "Expected Shortfall" (ES, aka CVaR), a coherent measure of risk
- 2000s VaR and ES are widely adopted by financial institutions as complementary tools
- 2013/2016: Basel Committee replaces $VaR_{1\%}$ with $ES_{2.5\%}$
  - VaR is maintained for model backtesting

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BASEL VAR BACKTEST: TRAFFIC LIGHT SYSTEM

Table 2

Zone	Number of exceptions	Increase in scaling factor	Cumulative probability
Green zone	0	0.00	8.11%
	1	0.00	28.58%
	2	0.00	54.32%
	3	0.00	75.81%
	4	0.00	89.22%
Yellow zone	5	0.40	95.88%
	6	0.50	98.63%
	7	0.65	99.60%
	8	0.75	99.89%
	9	0.85	99.97%
Red zone	10 or more	1.00	99.99%

- (Source: BCBS FRTB 2013) Same mechanism since 1996

CURRENT DEBATE

- VaR has serious shortcomings but allows for immediate backtesting

- Is it possible to backtest ES?
  – If not, can regulation be based on models that can't be validated?

- Backtesting means **model validation**
  – Fundamental for a risk system, non negotiable feature
    - Only way to say if your model is doing a good job
  – Key property for a regulatory standard
    - ES now adopted by "Basel 3" (in FRTB drafts, since 2013)
      - IAIS reached the opposite conclusion for ICS

- Backtesting also <u>means nothing</u>, until someone gives a formal definition
  – We attempt a definition and try to put some order

VAR AND ES

A closer look

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VAR DEFINITION AND BACKTESTING

- VaR can be defined as

  $VaR_{\alpha} = -F^{-1}(\alpha)$

$Pr[X < -VaR_{\alpha}] = \alpha$

- The probability for a loss to exceed VaR_{α} is exactly α
  - In a sequence of N independent correct predictions, the losses exceeding VaR_{α} are **binomial distributed with mean $N\alpha$**
    - **More (less) exceptions show under- (over-) estimation of risk**

- Notice the expression $\mathbb{E}[\alpha - (X + VaR_{\alpha} < 0)] = 0$
  - A null expected value involving only VaR and X

ES DEFINITIONS AND BACKTESTING DIFFICULTY

- Several equivalent ES definitions
  - $ES_\alpha = \int_0^\alpha VaR_p \, dp$
  - $ES_\alpha = -\frac{1}{\alpha} \mathbb{E}[X(X+VaR_\alpha<0) - VaR_\alpha (\alpha - (X+VaR_\alpha<0))]$

    $= VaR_\alpha - \frac{1}{\alpha} \mathbb{E}[(X+VaR_\alpha)(X+VaR_\alpha<0)]$
    - Reduces to $ES_\alpha = -\mathbb{E}[X \mid X<-VaR_\alpha]$ only for continuous cdf
  - $ES_\alpha = \min_v [v - \frac{1}{\alpha} \mathbb{E}[(X+v)(X+v<0)]]$ (Uryasev and Rockafellar)
    - $VaR_\alpha = \operatorname{argmin}_v [\cdots]$

- Notice that there's no way to build an expression like

$\mathbb{E}[f(ES_\alpha, X)] = 0$

– f will also depend on VaR_α

- We will see that basically ES is not backtestable for this simple reason

ELICITABILITY AND BACKTESTABILITY

Formalizing why a measure is backtestable or not

ELICITABILITY

- A statistic $Y(X)$ is said to be **elicitable** if it solves

$$Y(X) = \arg\min_y \mathbb{E}[S(y,X)]$$

  for some scoring function $S(y,x)$

- Popular examples:
  - Mean: $X = \arg\min_y \mathbb{E}[(y-X)^2]$
  - Median: $X = \arg\min_y \mathbb{E}[|y-X|]$
  - α-quantile: $q_\alpha = \arg\min_y \mathbb{E}[(X-y)(\alpha-(X-y<0))]$ → VaR !

- If a statistic Y is elicitable, given forecasts y_t and realizations x_t the mean score ranks models: the lower, the better

$$S = \frac{1}{T}\sum_{t=1}^{T} S(y_t, x_t)$$

MODEL VALIDATION VS MODEL SELECTION

- If a measure is elicitable, we can rank models by their mean score
  – relative, not absolute scale
  – A single mean score value, per se, tells nothing
- A mean score compares different predictions to the same process
  – Ex: Bank A wants to select the best in a class of VaR forecast models
  – Elicitability → Model selection: relative scale
- Backtesting requires absolute significance
  – Ex: Bank A wants to validate its model
  – Backtesting → Model validation: absolute scale

IDENTIFIABILITY

- A statistic $Y(X)$ is said to be **identifiable** if it solves

$$\mathbb{E}[I(y,X)]=0 \quad \text{when} \quad y=Y(X)$$

  for some identification function $I(y,x)$

- Popular examples:
  - Mean: $\mathbb{E}[X-X]=0$
  - Median: $\mathbb{E}[1/2-(X-X<0)]=0$ (cont. cdf's)
  - α-quantile: $\mathbb{E}[\alpha-(X-q_{\alpha}<0)]=0$ (cont. cdf's)

- Under regularity conditions, elicitability and identifiability coincide

$$S(y,x)=\int^{y} I(t,x)\,dt \qquad \text{(Osband's principle, 1985)}$$

- Notice that VaR is always elicitable, but identifiable only if the cdf is continuous

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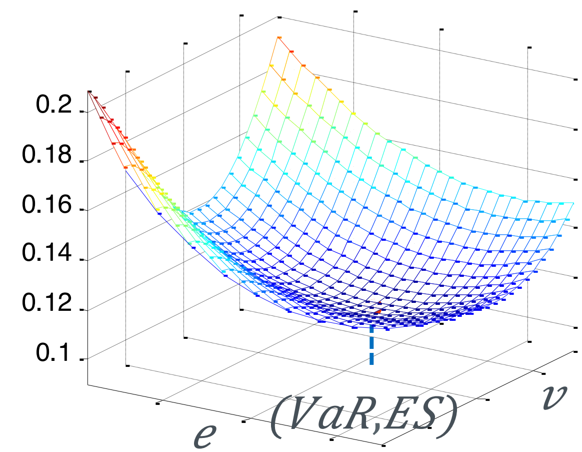
ES NEEDS VAR

- ES is neither elicitable nor identifiable per se (Gneiting, Ziegler, Weber, Bellini, …)
  - The pair (ES, VaR) however is both



- 2-Elicitability
  - Acerbi, Szekely 2014, $S^W(v,e,x)$
  - Ziegel, Fissler 2015, larger class of $S(v,e,x)$'s
- 2-Identification
  - Precise definition in Ziegel, Fissler 2015
  - From $ES_\alpha = VaR_\alpha - 1/\alpha \, \mathbb{E}[(X+VaR_\alpha)(X+VaR_\alpha<0)]$
    - $I_2(v,e,x) = \alpha(e-v) + (x+v)(x+v<0)$
- Striking parallel: also 'variance needs the mean', for the same properties

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BACKTESTABILITY – OUR DEFINITION

- **Definition**: a statistic $Y(X)$ is **backtestable** if there exists a **test function** $Z(y,x)$ such that

  $y \mapsto \mathbb{E}[Z(y,X)]$ is increasing

  $\mathbb{E}[Z(y,X)] \lesseqgtr 0$ when $y \lesseqgtr Y(X)$

- Very similar to identifiability, but not quite the same
  - We require more: monotonicity
  - We require less: we do not require $\mathbb{E}[Z(y,X)]=0$ in $y=Y(X)$

- No need for regularity conditions. VaR is always backtestable despite not always identifiable

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BACKTESTABLE RISK MEASURES

- It is immediate from $S(y,x)=\int^{y} Z(t,x)\,dt$ that
  - $Y(X)$ is backtestable iff it's elicitable with a y-convex scoring function S
    - The test function Z is the y-subdifferential of S
- All popular basic scoring functions happen to be already convex
  - VaR, Mean, Median, are backtestable
  - Expectiles are backtestable (Pandora's box)
- ES is not backtestable, because it's not elicitable
  - End of a long controversial debate

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HYPOTHESIS TESTING

- A **test statistic** of a backtestable risk measure is defined as

$$Z(y,x) = \frac{1}{T}\sum_{1}^{T} Z(y_t, x_t)$$

  - The H_0 (H_1) distribution of Z can always be resampled from predictive (resp. misspecified) distributions
  - Requires storage of predictive distributions day by day

- VaR is "super-backtestable" because the distribution of Z is model-independent, namely a binomial.
  - No need for resampling
  - Arguably a unique case (up to monotonic transformations)

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L’ES EST MORT, VIVE L’ES

How to backtest ES, nevertheless

HOW TO BACKTEST ES, NEVERTHELESS

- ES is 2-backtestable jointly with VaR
  - Follows immediately from its 2-identifiability
- This means that you can backtest ES only if you have previously backtested VaR, bilaterally, with success
- Generally speaking this seems a poor strategy
  - Hypothesis testing can rule out gross mistakes in VaR predictions, but never confirm that they are spot on!
- Questions
  - How sensitive is ES backtesting to VaR discrepancies?
  - How does a wrong VaR impact on the quality of ES backtest?

BACK TO BASICS

- We recall the expression valid for general distributions (Acerbi and Tasche 2001)

$$ES_{\alpha} = VaR_{\alpha} - \frac{1}{\alpha}\,\mathbb{E}[(X+VaR_{\alpha})(X+VaR_{\alpha}<0)]$$

- This yields a 2-identification/test function for VaR and ES

$$Z(v,e,x) = \alpha(e-v)+(x+v)(x+v<0)$$

- How sensitive is this function to VaR predictions?

BACK TO BASICS

- Now recall also Uryasev and Rockafellar's (2001) classical result

$$ES_{\alpha} = \min_{v} \left[v - 1/\alpha \, \mathbb{E}[(X+v)(X+v<0)] \right]$$

$$VaR_{\alpha} = \operatorname{argmin}_{v} [\cdots]$$

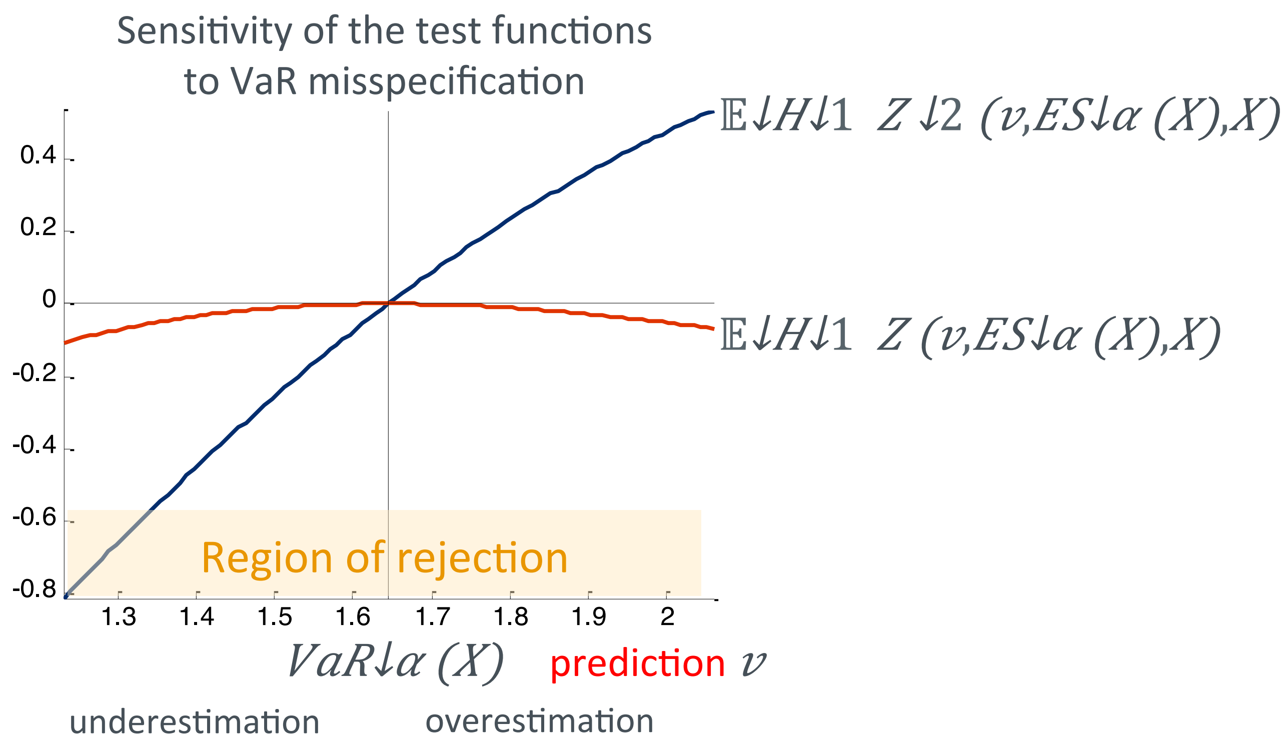
- Important consequence: if we backtest ES with $Z(v,e,x)$ then <u>any possible error in VaR prediction</u>
  1. Would make the backtest always <u>more prudential</u>
  2. Would have <u>no relevance at first order</u>

NEW PROPOSED TEST FOR ES

- $Z(v,e,x)=\alpha(e-v)+(x+v)(x+v<0)$
- Define $Z(\vec{X})=\sum_{t=1}^{T} \frac{Z(v_t, e_t, x_t)}{T\alpha e_t}$
- Hypotheses
  - $H_0: P_t^{[\alpha]} = F_t^{[\alpha]}$ *for all* t
  - $H_1: e_t \leq ES_{\alpha,t}^{F}$ for all t and $<$ for some t
    $v_t \sim VaR_{\alpha,t}^{F}$
- We have: $\mathbb{E}_{H_0}[Z]=0$ and $\mathbb{E}_{H_1}[Z]<0$
- VaR has to be tested separately, on two-sides, but loosely

SENSITIVITY TO VAR PREDICTION

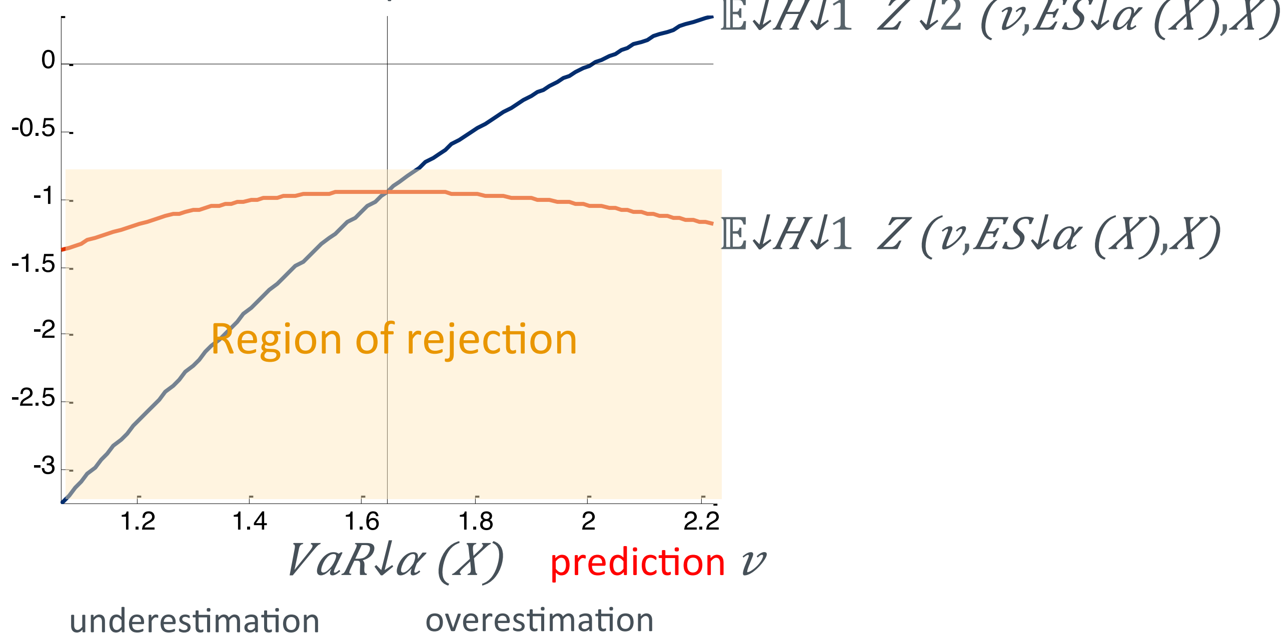
- Example: $\alpha=5\%$, true distributions are 250 i.i.d. N(0,1) returns
- **ES is estimated correctly**: $e=ES_\alpha(X)$



SENSITIVITY TO VAR PREDICTION

- Example: $\alpha = 5\%$, true distributions are 250 i.i.d. N(0,1) returns
- **ES is underestimated**: $e = ES_{\alpha}(X) - 1$

Sensitivity of the test functions
to VaR misspecification



CONSEQUENCES, IN PRACTICE

- The new test for ES lends itself to widespread adoption
  - It requires joint but mild validation of VaR
  - At worst, leads to a more penalizing test, in the case when VaR predictions should be completely wrong

- The test of ES requires nonetheless the day-by-day storage of predictive distributions for bootstrapping the $H\downarrow 0$ distribution of the test statistic and the computation of a p-value
  - VaR backtesting on the contrary requires storage of only one number per day, the prediction

- The method can be adopted for a traffic light system: it could be adopted in Basel regulation

DID VAR CREATE ES?

Indirect backtestability

STRIKING PARALLEL WITH VARIANCE

- Remember that also variance is 2-identifiable with the mean because of $\sigma^2 = \mathbb{E}[(X - X)^2]$

- Remember now that also in the case of variance we can express it as

$$\sigma^2 = \min_m \mathbb{E}[(X - m)^2]$$

- Exact same result on the 1st order irrelevance of misspecifications of the mean when backtesting the variance

- Coincidence?

- As a matter of fact, people have backtested variance for decades, without worrying much is the mean prediction was spot on

NOT A COINCIDENCE

- Suppose Y_1 backtestable: $Y_1(X) = \arg\min_y \mathbb{E}[S(y,X)]$

- Then $Y_2(X) = \min_y \mathbb{E}[S(y,X)] = \mathbb{E}[S(Y_1(X),X)]$ is automatically 2-backtestable with Y_1 from this very equation!
  - And tests of Y_2 based on this will be insensitive at 1^{st} order to Y_1 predictions and always biased in one direction only

- The couples (mean, variance) and (VaR, ES) are just two examples of this mechanism
  - Measures like Variance and ES are *indirectly backtestable*

- We can say that ES is generated by the backtestability of VaR!
  - VaR created its enemy

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CONCLUSIONS

CONCLUSIONS 1/2

- We provide a natural definition of backtestability which turns out to coincide with convex elicitability
- Most popular elicitable statistics happen to be convex too, hence backtestable. All non-elicitable statistics, notably ES, are not backtestable
- However the natural 2-identification function for VaR and ES has remarkable features
  - ES is 2-backtestable with VaR and the dependence of the ES test on VaR is zero at 1^{st} order and always of prudential type
  - We introduce a new test of ES based on these results that opens the way for valid practical backtests for ES

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CONCLUSIONS 2/2

- We notice that 2-backtestability of ES (and the above properties) is just an instance of a general situation which we term indirect backtestability, where a statistics is generated by the backtestability of another statistics.
  - Mean and variance are in the same exact relationship as VaR and ES
  - Variance and ES are just two instances of indirect backtestability

MODEL RISK KILLED MODELS

A short interlude

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BASEL TOURNAMENT

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AND THE WINNER IS ….

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THANKS FOR NOT FALLING ASLEEP

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APPENDIX

A non backtest of ES

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A NON-TEST OF ES

- Costanzino and Curran (2014) and Du and Escanciano (2015) propose an identical "backtest for ES"
- Observing that $ES_{\alpha} = \frac{1}{\alpha} \int_{0}^{\alpha} VaR_{q}\, dq$
- By *analogy*, they define "<u>failure rate of ES</u>" $\frac{1}{\alpha} \int_{0}^{\alpha} (X + VaR_{q} < 0)\, dq$
  - And show it has mean $= \alpha/2$; variance $= \alpha(4-3\alpha)/12$
- Problem:
  - The average of VaR failure rates is not the failure rate of the average VaR
  - The name is deceptive; it doesn't test ES
  - It's a test of the entire tail

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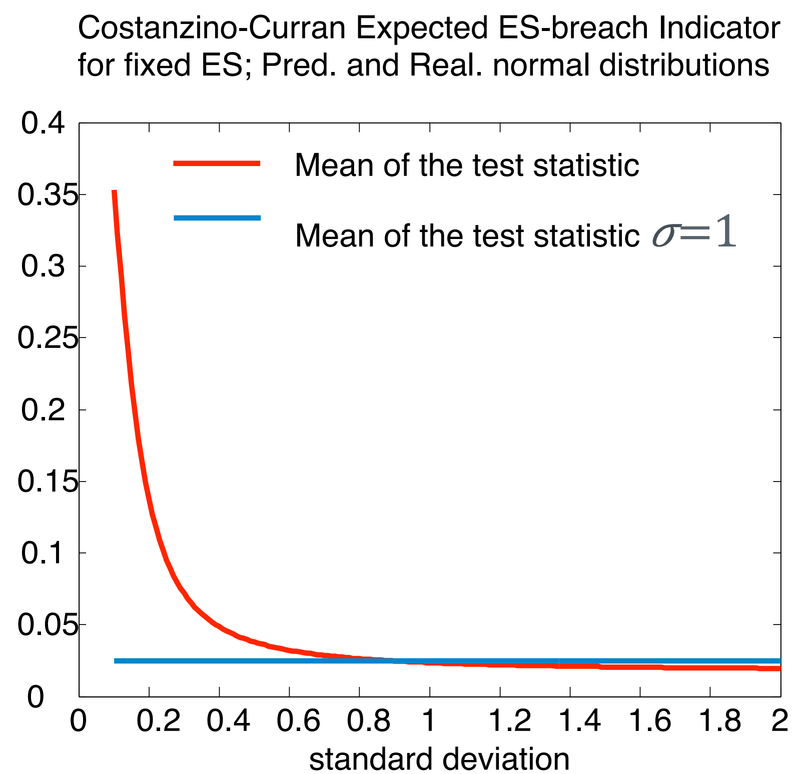
A NON-TEST OF ES

- Example:
  - $N(0,1)$ predictive distribution for ES_α, $\alpha=5\%$
  - $N(\mu(\sigma),\sigma^2)$, realized distribution
    - $\mu(\sigma)$ such that ES_α coincides with $N(0,1)$
- Prediction of ES_α perfect by construction
- We plot the expected value of the "failure rate of ES" as a function of σ
  - It is different from $\alpha/2=2.5\%$ for all $\sigma\neq1$
  - Shorter tails are penalized by the test and vice versa



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